Compressive Sensing for GPR Imaging

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Abstract—The theory of compressive sensing (CS) enables the reconstruction of sparse signals from a small set of non-adaptive linear measurements by solving a convex $\ell_1$ minimization problem. This paper presents a novel data acquisition and imaging algorithm for Ground Penetrating Radars (GPR) based on CS by exploiting sparseness in the target space, i.e., a small number of point-like targets. Instead of measuring conventional radar returns and sampling at the Nyquist rate, linear projections of the returned signal with random vectors are taken as measurements. Using simulated and experimental GPR data, it is shown that sparser and sharper target space images can be obtained compared to standard backprojection methods using only a small number of CS measurements. Furthermore, the target region can even be sampled at random aperture points.

I. INTRODUCTION

Ground Penetrating Radars (GPR) image subsurface structures by transmitting short electromagnetic pulses into the ground and processing the reflections [1]. The GPR transmissions spanning a region of interest form a synthetic aperture, whose impulse response is a spatially variant curve in the space-time domain. The total subsurface response, formed from a combination of the responses from all reflectors within the medium, can be inverted using a number of imaging algorithms; e.g., time-domain Standard Backprojection (SBP) [2], Fourier domain Synthetic Aperture Radar (SAR) image formation techniques [3], [4]. All of these algorithms require fine spatial sampling and Nyquist-rate times samples of the received signals. Hence, the data acquisition for GPR is the bottleneck of the general subsurface imaging process. In this paper, we propose a method that takes a very small set of informative measurements that still allow us to obtain the subsurface image.

Recent results in Compressive Sensing (CS) state that it is possible to reconstruct a $K$-sparse signal $x = \Psi s$ of length $N$ from $O(K \log N)$ measurements [5], [6]. CS takes non-traditional linear measurements, $y = \Phi x$, in the form of randomized projections. The signal vector, $x$, which has a sparse representation in a transform domain $\Psi$, can be reconstructed from $M = C (\mu^2 (\Phi, \Psi) \log N) K$ compressive measurements exactly with high probability by solving a convex optimization problem of the following form

$$\min \|x\|_1, \quad \text{subject to} \quad y = \Phi \Psi x.$$  \hspace{1cm} (1)

which can be solved efficiently with linear programming. Here $C$ is a small constant and $\mu(\Phi, \Psi)$ is the mutual coherence [5] between $\Phi$ and $\Psi$.

We use a basis pursuit strategy to formulate the GPR imaging problem as a dictionary selection problem where the dictionary entries are produced by discretizing the target space and synthesizing the GPR model data for each discrete spatial position. The assumption we make here is that the targets are point like reflectors at discrete spatial positions and the sparsity of the target space implies that the number of targets is much less than the total number of discrete spatial positions which depends on the spatial resolution used. The reason we use a point target reflectivity model is that the received data can be easily calculated for a point target. The point-like target assumption is not crucial; other models can be used as long as the targets are sparse or compressible in some transform domain, e.g., smooth targets are sparse in the Fourier or wavelet domain. If the received data can be calculated for other types of target models which are sparse in some domain, then CS ideas can still reconstruct the target space image.

CS captures the information about the sensed medium in a relatively small number of “random” measurements. Note that we do not take the random projections of the target space; instead, we only take random projections of the received signals at each aperture point. Then we formulate the CS reconstruction problem using our model(s) for the received signals as a function of target space positions. Our goal is not to reconstruct the received signals, but rather to create an image of the subsurface. This is done by solving an $\ell_1$ minimization problem, which is detailed in Section II.

II. THEORY: CS FOR GPR IMAGING

A. Creating a dictionary for GPR data

The GPR transmissions spanning a region of interest form a synthetic aperture, whose impulse response follows a spatially variant curve in the space-time domain. Although the response of targets might be more complex, we assume that the received signal at the receiver antenna reflected from a point target at $p$ is a time delayed and scaled version of the transmitted signal $s(t)$. Using the model the received signal at the GPR receiver antenna can be modelled as

$$z_i(t) = A s(t - \tau_i(p))$$  \hspace{1cm} (2)

where $\tau_i(p)$ is the total round-trip delay between the transmitter and the target at position $p$ for the $i$th aperture point, and $A$ is a scaling factor used to account for any spreading that occurs enroute. In our modelling of the received GPR signal
the parameter $\tau$ is very important and its calculation requires knowledge of wave velocities in both media as

$$\tau = \frac{d_1 + d_3}{v_1} + \frac{d_2 + d_4}{v_2},$$

where the distances $d_{1:4}$ are shown in Fig. 1.

![GPR measurement setup showing a bistatic GPR measurement scenario.](image)

Fig. 1. GPR measurement setup showing a bistatic GPR measurement scenario.

Standard imaging algorithms do a matched filtering of the measured data with the impulse response of the data acquisition process for each point of the space. Time domain standard backprojection (SBP) [2], [7], [8] can be formulated as follows.

$$f(x_n, y_n, z_n) = \int \int d(u_x, u_y, t) \delta(t - \tau(u_x, u_y, z_n)) du_x du_y,$$

where $d(u_x, u_y, t)$ is the measured GPR space-time data, $f(x_n, y_n, z_n)$ is the created subsurface image, and $\tau(u_x, u_y, z_n)$ is the total time delay from the antenna to the imaging point $(x_n, y_n, z_n)$ and back to the receiver. A weighting function on the aperture can also be used. Frequency domain imaging algorithms [3], [4] has a similar match filtering operation.

A dictionary for the GPR data can be created by discretizing the target space and synthesizing the GPR model data for each discrete spatial position. The target space $\tau_F$ lies in the product space $[x_f, y_f] \times [z_f, z_f]$, which must be discretized to generate the target space dictionary. Here $x_1, y_1, z_1$ and $x_f, y_f, z_f$ denote the initial and final positions of the target space to be imaged along each axis. Discretization generates the set of target points $B = \{x_1, \pi_2, \ldots, \pi_N\}$, where $N$ determines the resolution and each $\pi_j$ is a 3D vector $[x_j, y_j, z_j]$. Finally, we define the vector $b$ to be a weighted indicator function, i.e., a non-zero positive value at index $j$ if $b$ selects a target at $\pi_j$.

Using (2) and (3) the signal at the GPR antenna can be calculated for a given element of $B$. This allows us to write a linear relation between the target space indicator $b$ and the measured data at aperture $t$ as

$$\zeta(t) = \Psi b$$

where

$$\zeta(t) = \left[\zeta_1(t_0), \zeta_2(t_0 + \frac{1}{F_s}), \ldots, \zeta_N(t_0 + \frac{N_t - 1}{F_s})\right]^T,$$

with $F_s$ is the sampling frequency, $t_0$ is the appropriate initial time, and $N_t$ is the number of data samples. In (5), the $j^{th}$ column of $\Psi_j$ is

$$[\Psi_{ij}] = \frac{s(t - \tau_1(j))}{\|s(t - \tau_1(j))\|_2},$$

where $\Psi_j$ with (7) each column has unit norm and the spreading factor $A$ in (2) doesn’t need to be known. Only the time delay needs to be calculated. The matrix $\Psi$ is the dictionary (or, sparsity basis) corresponding to all discretized target points $B$ when the GPR is at the $i^{th}$ aperture point.

B. Compressive Sensing Data Acquisition

Standard GPR receivers sample the received signal at a very high rate $F_s$ as in (6). In compressive sensing (CS) rather than sampling $\zeta_i$ at $F_s$ we measure linear projections of $\zeta_i$ onto a second set of basis vectors $\Phi_{i,m}$, $m = 1, 2, \ldots, M$ which can be written in matrix form for the $i^{th}$ aperture point as

$$\beta_i = \Phi_i \zeta_i = \Phi_i \Psi_i b,$$

where $\Phi_i$ is an $M \times N_i$ measurement matrix. Fewer samples than the size of $b$ are taken. In the spirit of CS a small number of “random” measurements carry enough information about the signal. In (8) the matrix $\Phi_i$ can be selected to minimize the mutual coherence between $\Psi_i$ and $\Phi_i$. The CS data acquisition at a single aperture point is shown in Fig. 2.

![Data Acquisition for GPR at one single aperture point, (b) One possible compressive sensing implementation at the GPR receiver.](image)

Fig. 2. (a) Data Acquisition for GPR at one single aperture point, (b) One possible compressive sensing implementation at the GPR receiver.

The GPR is not directly measuring $\zeta_i$ but only taking inner products of it with the rows of the $\Phi_i$ matrix. The inner product operation can be implemented as shown in Fig. 2(b). Depending on the structure of $\Phi_i$, other implementations can also be used [9], [10].

C. GPR Imaging with Compressive Sensing

The result of the CS theory is that the target space indicator vector $b$ can be recovered exactly from $M = C \mu^2(\Phi, \Psi) \log N$ K CS measurements $\beta$ with overwhelming probability [5], by solving an $\ell_1$ minimization problem as

$$\text{minimize} \quad \|b\|_1 \quad \text{subject to} \quad \|\Phi_i b - \beta_i\|_2 < \epsilon,$$

where $\epsilon$ is a small positive constant.
where \( \rho(\Phi, \Psi) \) is the coherence between \( \Phi \) and \( \Psi \) defined as in [5]. The notations are \( \beta = [\beta_1^T, \ldots, \beta_L^T]^T \), \( \Psi = [\Psi_1^T, \ldots, \Psi_L^T]^T \), and \( \Phi = \text{diag}(\Phi_1, \ldots, \Phi_L) \).

The optimization problem in (9) is valid for the noiseless case. In general, the GPR signal is noisy, i.e., \( \zeta_i(t) = \zeta_i + n_i(t) \). Then the compressive measurements \( \beta_i \) at the \( i \)th aperture position have the following form:

\[
\beta_i = \Phi_i \zeta_i = \Phi_i \Psi_i b + u_i
\]

where \( u_i = \Phi_i n_i \sim \mathcal{N}(0, \sigma_n^2) \) and \( n_i \) is the concatenation of the noise samples at aperture point \( i \) which is assumed to be \( \mathcal{N}(0, \sigma_n^2) \). Since \( \Phi_i \) is deterministic, we have \( \sigma^2 = (\sum_{m=1}^N \phi_{im}^2) \sigma_n^2 \). Hence, if we constrain the norm of the \( \phi_{im} \) vectors to be one, then \( \sigma^2 = \sigma_n^2 \).

It is shown in [11], [12] a stable recovery of the sparsity pattern vector \( b \) is possible by solving the following convex optimization problem

\[
b = \arg\min \| b \|_1 \quad \text{s.t.} \quad \| A^T(\beta - Ab) \|_\infty < \epsilon = \epsilon_N \sigma,
\]

where \( A = \Phi \Psi \). Selecting \( \epsilon_N = \sqrt{2\log N} \) makes the true \( b \) feasible with high probability. The optimization problems in (9) and (11) both minimize convex functionals, so a global optimum is guaranteed.

### III. RESULTS

A test example will illustrate the ideas presented in the previous section. A 2D slice of the target space with 30 cm x 30 cm dimensions containing three randomly placed point targets is taken. The target space image is shown in Fig. 3(a).

To simulate the space-time domain response of the target space, a GPR simulated data generation program is developed in MATLAB. System specifications such as aperture size, antenna height, transmitter-receiver distance, target depth, as well as soil type and conductivity are all adjustable parameters of the program. Targets are simulated as point targets. For this example a bistatic antenna pair with antenna height 10 cm and transmitter-receiver distance 5 cm and a dry soil with permittivity \( \varepsilon = 4 \) is used.

The noisy space-time domain response of the target space is shown in Fig. 3(b). The signal-to-noise ratio (SNR) for this example is 8.6 dB. Instead of measuring the space-time domain response at each aperture position, inner products of the time-domain response with rows of a random matrix \( \Phi_i \) of size 10 x 512 with entries independently drawn from \( \mathcal{N}(0, 1/\sqrt{512}) \) are measured. Ten inner product measurements are done at each aperture point making 300 measurements in total for 30 aperture points instead of 512 x 30 raw space-time domain measurements (Fig. 3(c)). Note that 300 measurements are the only information we have about the target space area. The number of targets is not assumed to be known.

The sparsity pattern vector for the target space has length 900 and we have 300 measurements which gives an underdetermined system of equations, \( \beta = Ab \). One possible feasible result is the least squares solution, \( \hat{b} = A^T(AA^T)^{-1}\beta \). The target space image for this is shown in Fig. 3(d). Although the solution is feasible in the sense that it satisfies the compressive measurements it gives no sensible information about possible target positions. Using the sparsity of the target space information and solving the convex optimization problem defined in (11) gives the image shown in Fig. 3(e). The result from Fig. 3(d) is used as a feasible starting solution for convex optimization. The \( \ell_1 \)-magic package [13] is used for convex optimization programming. It can be seen that the actual target positions are found correctly and the obtained image is sparse compared to the standard backprojection result (Fig. 3(f)) which is the image obtained if we had measured the whole space-time response. Both of the images in Fig. 3(e,f) are normalized to their own maxima and are shown on the same 60-dB scale. The convex optimization result has less clutter in the image since the problem forces a sparse solution through the \( \ell_1 \) norm minimization.

A. Effect of Measurement Matrix \( \Phi \)

The results shown in Fig. 3 are created with a selection of a single random measurement matrix \( \Phi \). The effect of
the random measurement matrix on the reconstructed target space is tested with a Monte Carlo simulation. At each time an independent random measurement matrix is selected using a new seed for the random number generator and the target space image is reconstructed solving the convex optimization problem in (9). To only test the effect of random measurement matrix a noiseless simulated GPR data set is used. It is observed that any random measurement matrix works equally well given that the minimal measurement number requirement is satisfied.

Previous example uses a measurement matrix whose entries are drawn from a normal distribution. It is seen that choosing the entries of $\Phi$ as $\pm 1$ with equal probability also works equally well. Choosing different type measurement matrices and their effects on the algorithm performance will be evaluated in a future work.

B. Random Spatial Sampling

The convex optimization problem in (11) solves for the target space using measurements from different aperture positions jointly. Satisfying the minimal required total measurement number for correct reconstruction, the used spatial aperture positions can be reduced. Figure 4 shows the results from randomly sampled spatial positions. In Fig. 3 ten compressive measurements at 30 spatial positions were used. This example uses 15 randomly selected aperture positions out of a total of 30 and at each aperture point 20 compressive measurements are taken. The actual target space is the same as Fig. 3(a). The space-time domain response is shown in Fig. 4(a). The skipped aperture positions are seen with black stripes. The compressive measurements at the selected aperture positions are shown in Fig. 4(b). Convex optimization and standard backprojection results are compared as 60-dB scaled images in Figs. 4(c) and (d). Although the CS result is a little cluttered compared to using full aperture, it is significantly better than the standard backprojection algorithm which uses the space-time domain data in Fig. 4(a).

The important point here is that the proposed method is robust to sampling less in spatial domain. Compared to the full aperture results from Fig. 3(c) and (l), randomly sampled aperture affected the backprojection result worse than the proposed method. This can be observed from the higher artifact level in Fig. 4(d). We think that the proposed method is more robust to random spatial sampling in the GPR measurements than the standard match-filtering based imaging algorithms.

C. Experimental Results

The results in Figs. 3 and 4 assume that the targets are point reflectors and the medium is homogeneous with known constant propagation speed. The effects of modelling errors like non-point targets, discretization of the target space and velocity mismatches don’t take role in previous examples. Detailed analysis of these effects will be discussed in a future paper due to lack of space here.

This section presents CS imaging of a 1” diameter metal sphere held in the air at a height of 36.5 cm on a styrofoam support. The experimental setup is shown in Fig. 5(a). Since the GPR antenna and the target are in air, the wave speed is known to be $c = 3 \times 10^8$ m/s; but there are non-idealities such as the target not being a point target, the measured data having non-gaussian noise and the target not being exactly at the discrete spatial points of the dictionary $B$. The initial results on this experimental data show that CS can handle these kinds of non-ideal situations.

The GPR antenna has a multistatic configuration [14] but for the results shown in this section only the data from the T1-R1 bistatic pair is used. The raw data measured over the target for a 2D slice is shown in Fig. 5(b). Since the CS measurement system is not yet built in hardware, standard space-time domain samples are obtained and the compressive measurements are created offline. Ten measurements are used at each of the 70 aperture points which makes a total of 700 CS measurements. Two different case of $\Phi_i$ are tested, Fig. 5(c) and (d) shows the used compressive measurements when different and same $\Phi_i$ is used at each aperture $i$ respectively. Using the same measurement matrix at each aperture will save from the memory requirements and will be much easier to implement.

For CS target space reconstruction, the Dantzig Selector (11) is used with $\epsilon = 0.5||A^T \beta||_\infty = 1.34 \times 10^{-4}$ for measurement sets shown in Fig. 5. The result of the Dantzig Selector for Fig. 5(c) and (d) are shown in Fig. 6(a) and (b) respectively. Note that the target is an” sphere where its bottom is positioned at 36.5 cm height. While both measurement sets can construct a sparse representation of the target space successfully, it is seen that using different $\Phi_i$ at each aperture has the potential of extracting more information about the target space for the same $\epsilon$ level. The results of the proposed algorithm is compared with the standard backprojection algorithm.
which uses the whole space-time domain data (Fig. 5(b)). Figure 6 shows that proposed method does a much better job at creating a sharper and sparser target space image than the backprojection result shown in Fig. 6(c). Note that while backprojection uses a total of 15,400 standard time domain samples, CS uses only 700 compressive measurements. All images are shown on a 30-dB scale and are normalized to their own maxima.

IV. CONCLUSIONS

A novel data acquisition and imaging algorithm for Ground Penetrating Radars based on compressive sensing is demonstrated. The new method exploits prior knowledge of the sparseness in the target space. An $\ell_1$ minimization convex optimization problem is solved with a small number of random compressive measurements taken at randomly chosen aperture points to reconstruct the target space image. Initial results from simulated and experimental GPR data shows that extremely sparser images can be obtained with the proposed method compared to standard backprojection imaging algorithms.

REFERENCES