An Investigation of Numerical Dispersion in the Vector Finite Element Method Using Quadrilateral Elements

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Abstract—The discretization inherent in the vector finite element method results in the numerical dispersion of a propagating wave. The numerical dispersion of a time-harmonic plane wave propagating through an infinite, two-dimensional, vector finite element mesh composed of uniform quadrilateral elements is investigated in this work. The effects on the numerical dispersion of the propagation direction of the wave, the order of the polynomials used for the basis functions, and the electrical size of the elements are investigated. Simple formulas are presented which are excellent approximations to the exact numerical dispersion. The numerical dispersion is validated by a numerical example.

I. INTRODUCTION

The vector finite element method is becoming an increasingly popular technique in computational electromagnetics for solving vector field problems [1]-[4]. The main reason for this is that it does not suffer from spurious or nonphysical solutions for many types of problems as does the node based finite element method [5]. In addition, boundary conditions are generally easier to impose along conductor edges and material interfaces when vector finite elements are used.

Even though the vector finite element method has been extensively used, the errors associated with it have not been thoroughly investigated. The errors must be quantified in order for one to have complete confidence in the numerical solution. For finite elements, one of the most significant errors arises from the inability of the polynomial basis functions to represent the fields exactly within an element. A wave propagating through a finite element mesh will experience numerical dispersion as a result of this error. Many researchers have looked at the numerical dispersion of plane waves in the node based finite element method for the two-dimensional case. Mullen and Belytschko investigated the numerical dispersion of a number of first-order uniform triangular elements and quadrilateral elements [6]. They indicated that the dispersion depends on the node density and the direction of propagation through the mesh. In addition, they considered the effects of mass lumping (diagonalization) on the dispersion. Lynch et al. demonstrated the effect of lossy materials on the numerical dispersion for first-order uniform triangular and rectangular meshes [7]. Recently, Lee and Cangellaris extended the analysis to second-order quadrilateral elements for a plane wave propagating along a mesh axis [8].

In two-dimensional problems, several types of vector elements are used. One of the most common, and of interest here, is the quadrilateral element. The numerical dispersion of a time-harmonic plane wave propagating through an infinite, two-dimensional, vector finite element mesh composed of uniform quadrilateral elements is investigated in this work. This numerical dispersion can be characterized by a cumulative phase error. The phase error is quantified as a function of the electrical size of the elements, the direction of propagation of the plane wave, and the order of the elements. It is shown that the phase error for the first- and second-order, quadrilateral, vector finite elements is the same as that for the corresponding nodal elements. Simple formulas are presented which are excellent approximations to the exact numerical dispersion. Finally, numerical results that verify the analysis are given.

II. VECTOR FINITE ELEMENT FORMULATION

In order to quantify the numerical dispersion of the vector finite element method, consider an infinite, linear, two-dimensional, homogeneous, isotropic, and source free region. The field in this region is governed by the vector Helmholtz equation:

$$\nabla \times (\nabla \times \vec{E}) - k^2 \vec{E} = 0$$  \hspace{1cm} (1)

where an $e^{j\omega t}$ time dependence is assumed and $k = \omega \sqrt{\mu \varepsilon}$ is the wavenumber. In a finite region, the following boundary conditions are used for the finite element formulation [9]:

$$\hat{n} \times \vec{E} = 0$$  \hspace{1cm} (2)

on the portion of the boundary $\Gamma_1$ that lies on a perfect electric conductor and

$$\hat{n} \times (\nabla \times \vec{E}) + \gamma \hat{n} \times (\vec{n} \times \vec{E}) = \vec{U}$$  \hspace{1cm} (3)

which is used on all other boundaries $\Gamma_2$. The role of this boundary condition is determined by the values of $\gamma$ and $\vec{U}$. In the finite element method, the region of interest is subdivided into discrete elements. Then, the vector field is approximated within an element by an expansion in terms of vector basis
functions:

$$\tilde{\mathbf{E}}^e = \sum_{i=1}^{n} \tilde{N}_i^e \mathbf{E}_i$$  \hspace{1cm} (4)

where $\tilde{N}_i^e$ are the vector basis functions, $\mathbf{E}_i^e$ are the unknown coefficients, and $n$ is the number of unknown coefficients. A system of linear equations is obtained by Galerkin’s method. The weighted residual for the $e$th element is [9]

$$R_j^e = \int_{\Omega_e} \left[ \nabla \times \left( \nabla \times \tilde{\mathbf{E}}^e \right) - k^2 \tilde{\mathbf{E}}^e \right] d\Omega$$  \hspace{1cm} (5)

where $\Omega_e$ is the area enclosed by the element $e$. Substituting the field expansion into the residual equation and placing it in the “weak form” gives the residual equation for the $e$th element:

$$R_j^e = \sum_{i=1}^{n} E_i^e \left[ \int_{\Omega_e} \left( \nabla \times \tilde{N}_j^e \right) \cdot \left( \nabla \times \tilde{N}_i^e \right) d\Omega \right] - k^2 \int_{\Omega_e} \tilde{N}_j^e \cdot \tilde{N}_i^e d\Omega + \gamma \int_{\Gamma_e} \left( \nabla \times \tilde{N}_j^e \right) \cdot \left( \nabla \times \tilde{N}_i^e \right) d\Gamma + \int_{\Gamma_e} \tilde{N}_j^e \cdot \tilde{U} d\Gamma$$  \hspace{1cm} (6)

where $j = 1, 2, 3, \ldots, n$.

The global equations are obtained by expanding the above equation and then assembling the local equations for all the elements. The resulting equations are set equal to zero, and the system of linear equations is solved to obtain the unknown coefficients $\mathbf{E}_i^e$. A more detailed explanation is given in [9].

### III. Dispersion Analysis

Consider a plane wave propagating through an infinite uniform mesh at an arbitrary angle $\phi$ in the $x$-$y$ plane. It is well known that a plane wave is an exact solution to the vector Helmholtz equation (1). For this solution, it is easy to show that the field at any point $p$ is related to the field at any other point $q$ by a simple phase factor:

$$\tilde{\mathbf{E}}_q = \tilde{\mathbf{E}}_p e^{-j k \tilde{x} \cdot \Delta \tilde{r}}$$  \hspace{1cm} (7)

where $k = \cos \phi \tilde{a}_x + \sin \phi \tilde{a}_y$ is a unit vector pointing in the direction of propagation and $\Delta \tilde{r}$ is the vector from point $p$ to point $q$.

A plane wave propagating along the mesh at the angle $\phi$ is a solution to the discretized vector Helmholtz equation. The plane wave propagates with a numerical wavenumber $k$. Consider the points $p$ and $q$ depicted in Fig. 1. The relationship between the field at any point $p$ within an element and any other point $q$ at the same relative position within another element is a simple phase factor:

$$\tilde{\mathbf{E}}_q = \tilde{\mathbf{E}}_p e^{-j k \tilde{x} \cdot \Delta \tilde{r}}$$  \hspace{1cm} (8)

where $\Delta \tilde{r} = m \Delta \tilde{r}_x + n \Delta \tilde{r}_y$ is the vector from the point $p$ to point $q$ in which $m$ and $n$ are restricted to be integers.

However, the relationship between the fields at two arbitrarily located points ($m$ or $n$ is not an integer) is not as simple.

#### A. First-Order Element

The basis functions used in vector finite elements are of mixed order. In this paper, two different types of elements are considered. The first is commonly called the edge element and is referred to here as the first-order element. In this element the vector basis functions are constant in the direction of the vector component and linear in the direction transverse to it. The element is depicted in Fig. 2(a) and the vector basis functions, $\tilde{N}_i^e$, are the usual first-order vector finite element basis functions [9]. Note that for this element, the unknowns $E_i^e$'s correspond to the constant value of the field component tangent to the edge.

Fig. 2(b) shows a local portion of an infinite mesh of first-order, two-dimensional, vector finite elements. Two types of edges occur in this mesh, an $x$-directed edge and a $y$-directed edge. The relationship between the field at the two types of edges is not given by (8) because the edges are not at the same relative element position in the mesh. Thus, it is necessary to consider an equation for the unknown associated with a $x$-directed edge and the unknown associated with a $y$-directed edge to derive the dispersion relation. The necessary equations can be obtained by setting the total residual at these edges equal to zero. The equations for the total residual at edges $i$ and $i+6$ are

\[
\begin{align*}
2 \left( 1 - \frac{k^2 h^2}{3} \right) E_i + E_{i+5} + E_{i+9} - E_{i+6} - E_{i+8} &= 0 \\
- \left( 1 + \frac{k^2 h^2}{6} \right) (E_{i+1} + E_{i+2}) &= 0 \hspace{1cm} (9)
\end{align*}
\]

\[
\begin{align*}
2 \left( 1 - \frac{k^2 h^2}{3} \right) E_{i+6} + E_{i+4} + E_{i+3} - E_{i+1} &= 0 \\
- \left( 1 + \frac{k^2 h^2}{6} \right) (E_{i+4} + E_{i+5}) &= 0 \hspace{1cm} (10)
\end{align*}
\]

where $h = \Delta x = \Delta y$ is the length of an edge.

From (8), the relationship between unknowns associated with the $x$-directed edges is

$$E_{\ell}(\Delta \tilde{r}) = E_\ell e^{-j k \tilde{x} \cdot \Delta \tilde{r}} \hspace{1cm} \ell = i + 1, i + 2, i + 3, i + 4$$  \hspace{1cm} (11)
where \( \Delta \vec{r} \) is the vector that points from the center of the edge \( i \) to the center of edge \( \ell \). The relationship between the unknowns associated with the \( y \)-directed edges is

\[
E_\ell(\Delta \vec{r}) = E_{i+6} e^{-jkh \Delta \vec{r}} \quad \ell = i + 5, i + 7, i + 8, i + 9
\]

where \( \Delta \vec{r} \) is the vector that points from the center of the edge \( i + 6 \) to the center of edge \( \ell \). By substituting (11) and (12) into (9) and (10) and eliminating \( E_i \) and \( E_{i+6} \), an expression for \( k \) in the form of a transcendental equation can be obtained:

\[
(k^2 h^2 + 12 \cos k_x \cos k_z + 4(k^2 h^2 - 6) = -2(k^2 h^2 + 3)(\cos k_x + \cos k_z)
\]

where \( k_x = \frac{k}{h} \sin \phi \) and \( k_z = \frac{k}{h} \cos \phi \). This dispersion relation can be shown to be the same as the dispersion relation for the first-order, two-dimensional (quadrilateral), nodal finite element.

The phase error that results from the dispersion, expressed in degrees per wavelength, is defined as

\[
\delta_p = 360 \left| \frac{k - \bar{k}}{k} \right|.
\]

Fig. 3 is a graph of the phase error as a function of \( \lambda/h = 2\pi/kh \) (dashed lines) for selected values of \( \phi \). As expected, the error becomes smaller as the electrical size of the element is decreased. The error is seen to be proportional to \( (h/\lambda)^2 \) for large values of \( \lambda/h \); thus, \( \bar{k} \) converges at the rate \( O\left(\frac{h}{\lambda}^2\right) \).

Fig. 4 is a polar graph of the phase error for selected values of \( \lambda/h \) as a function of \( \phi \) (dashed lines). Note that, due to the symmetry, the same dispersion error is obtained at the angles \( \phi, \phi + n 90^\circ \), and \( m 90^\circ - \phi \), where \( m \) and \( n \) are integers. From this graph, it is apparent that the smallest error occurs when the plane wave advances the elements diagonally \( (\phi = \pm 45^\circ, \pm 135^\circ) \) and the largest error occurs when the wave is propagating along an axis of the mesh i.e., \( (\phi = 0^\circ, \pm 90^\circ, 180^\circ) \). In the case of \( \phi = 0^\circ (90^\circ) \) the dispersion relation simplifies to become

\[
\bar{k}(\phi = 0, kh) = \frac{1}{h} \cos^{-1} \left[ \frac{1 - (kh)^2/3}{1 + (kh)^2/6} \right].
\]

The above equation is the same as the dispersion relation for the first-order one-dimensional nodal element [8]. This is because at this incidence angle the plane wave field only has a component in the \( y(x) \) direction which is approximated by a linear function in \( x(y) \). For \( \lambda/h > 5 \), a good approximation to the dispersion relation (13) is

\[
\bar{k}(\phi, kh) = \bar{k}(0, kh) + \frac{1}{2} \left[ \bar{k}(0, kh/\sqrt{2}) \right. \\
- \bar{k}(0, kh) \left[ 1 - \cos (4\phi) \right].
\]

The solid lines in Figs. 3 and 4 are the phase error calculated using this approximation. The approximation is seen to be excellent; the maximum error is 0.07 percent at \( \lambda/h = 5 \).
B. Second-Order Element

The second type of element is referred to as the second-order element. In this element, the vector basis functions are linear in the direction of the vector component and quadratic in the transverse direction [10]. The element is depicted in Fig. 5(a) and the vector basis functions, \( N_i^j \) are given in the appendix. As can be seen, this element is considerably more complicated than the first-order element. To simplify things, the first step in the direction of the vector component and quadratic in the transverse direction \([lo]\). The element is depicted in Fig. 5(a) associated with an edge. Fig. 5(b) shows a local portion of an infinite mesh of simplified second-order elements. As in the case with the first-order element mesh, there are two types of edges. However, there are now two unknowns associated with each edge instead of only one. The relationship between the field at the two types of edges is not given by (8); neither is the relationship between the field at the two unknowns associated with each edge. As a result, it is necessary to consider equations for the two unknowns associated with a \( x \)-directed edge and the two unknowns associated with a \( y \)-directed edge to derive the dispersion relation. The residual equations for the unknowns associated with \( i + 4 \), \( i + 5 \), \( i + 11 \), and \( i + 14 \) are computed. From (8), the relationship between the unknowns along the \( x \)-directed edges are

\[
E_\ell (\Delta \tau) = E_{i+4} e^{-jkk \Delta \tau} \\
\ell = i, i + 2, i + 6, i + 8 \tag{17}
\]

where \( \Delta \tau \) is the vector that points from a position on edge \( i + 4 \) or \( i + 5 \) to the corresponding position on edge \( \ell \). The relationship between the unknowns along the \( y \)-directed edges are

\[
E_\ell (\Delta \tau) = E_{i+5} e^{-jkk \Delta \tau} \\
\ell = i + 1, i + 3, i + 7, i + 9 \tag{18}
\]

where \( \Delta \tau \) is the vector that points from a position on edge \( i + 11 \) or \( i + 14 \) to the corresponding position on edge \( \ell \). Substituting the relationships into the four residual equations results in the following set of equations:

\[
\begin{align*}
& u \left( \frac{1}{v} + v \right) c_1 + \left( \frac{1}{v} + v \right) c_2 + \left( \frac{u}{v} - \frac{v}{u} \right) c_3 \\
& + \left( r + \frac{v}{u} - t - ut \right) c_4 + n(t - r) c_5 + 2uc_6 + 2uc_7 = 0 \tag{21}
\end{align*}
\]

\[
\begin{align*}
& u \left( \frac{1}{v} + v \right) c_1 + r \left( \frac{1}{v} + v \right) c_2 + u \left( \frac{v}{u} - \frac{u}{v} \right) c_3 \\
& + \left( u + v - \frac{u}{v} - us \right) c_4 + (s - 1) c_5 + 2utc_6 + 2uc_7 = 0 \tag{22}
\end{align*}
\]

Fig. 6 is a graph of the phase error as a function of the electrical size of the element \( \lambda/h \) (dashed lines) for selected values of \( \phi \). The second-order element, like the first-order element, has an error that decreases as the elements decrease in electrical size. However, the error is seen to decrease at a quicker rate; the rate of convergence of \( \hat{k} \) is seen to be \( O \left( (h/\lambda)^4 \right) \) instead of \( O \left( (h/\lambda)^2 \right) \). Fig. 7 shows the phase error as a function of \( \phi \) (dashed line) for selected values of \( \lambda/h \). The second-order element, like the first-order element, has an error that is smallest when \( \phi = \pm 45^\circ, \pm 135^\circ \) and is largest when \( \phi = 0^\circ, 90^\circ, 180^\circ \); however, the ratio of the
The largest error to the smallest error is greater for the second-order element. In addition, the error for the second-order element displays the same type of symmetry as the does the error for the first-order element. In the case of $\phi = 0^\circ$ (90$^\circ$), the dispersion relation simplifies to become
\begin{equation}
k(\phi = 0, kh) = \frac{1}{2h} \cos^{-1} \left( \frac{15 - 26k^2h^2 + 3k^4h^4}{15 + 4k^2h^2 + k^4h^4} \right)
\end{equation}
which is the same as the dispersion relation for the second-order, one-dimensional, nodal element and the second-order, two-dimensional (quadrilateral), nodal element with $\phi = 0^\circ$ (90$^\circ$). This is because at this incidence angle the plane wave field only has a component in the $y(z)$ direction which is approximated by a quadratic function in $x(y)$. For $\lambda/h > 5$, a good approximation to the dispersion relation represented by the four transcendental equations is also given by (16) using $\hat{k}$ from (26). The solid lines in Figs. 6 and 7 are the phase error calculated using this approximation. Once again the approximation is seen to be excellent; the maximum error is 0.05 percent at $\lambda/h = 5$.

IV. NUMERICAL VERIFICATION

A vector finite element code was written to numerically verify the dispersion analyses and to show that they have relevance for a finite size mesh. The geometry modeled was an infinitely wide, parallel-plate waveguide of width $a$ and length $\ell$ with perfectly conducting walls, as in Fig. 8. The phase constant of the $TM_1$ mode in the waveguide is calculated first by the vector finite element code and then it is predicted using the dispersion relations. By comparing the results, it is seen that the dispersion relations are correct.

The arrangement of elements used in the code is depicted in Fig. 8. Due to the symmetry of the $TM_1$ mode, only one half of the waveguide is divided up into elements and a perfect magnetic conductor (PMC) is placed along the middle of the waveguide. The $TM_1$ mode is injected at $x = 0$ and is absorbed at $x = \ell$ using the appropriate Neumann boundary conditions. The error, $\delta_\beta$, in the numerical phase constant, $\beta$, in degrees per guide wavelength is
\begin{equation}
\delta_\beta = 360 \left| \frac{\beta - \beta}{\beta} \right|
\end{equation}
where $\beta$ is the phase constant of the $TM_1$ mode. Fig. 9 is a graph of the error as a function of $a/h$ for first and second order elements for selected values of $a/h$ (symbols).

The numerical phase constant is also predicted using the dispersion relations by viewing the $TM_1$ mode as the superposition of two numerical plane waves, one that propagates in the $+\phi$ direction and another that propagates in the $-\phi$ direction, both with a numerical wavenumber $\hat{k}$. For the $TM_1$ mode, $\hat{k}$ and $\phi$ are related:
\begin{equation}
\frac{\pi}{a} = \hat{k} \sin \phi .
\end{equation}
The dispersion relations along with (28) are solved simultaneously to obtain $\hat{k}$ and $\phi$. Then, the numerical phase constant is obtained from the relation
\begin{equation}
\hat{\beta} = \hat{k} \cos \phi .
\end{equation}
The error in the numerical phase constant derived from the dispersion relations are also graphed in Fig. 9. The agreement between the results from the code and the dispersion relations is seen to be excellent.

V. CONCLUSION

The phase error that results from the discretization error in the vector finite element method when quadrilateral elements are used has been quantified. The phase error for the first- and second-order, quadrilateral, vector finite elements is the same as that for the corresponding nodal elements [11]. For both the nodal and vector elements, this error is a function of the propagation direction, the electrical size of the element, and the order of the element. In addition, Figs. 3 and 6 show that $\tilde{k}$ for a $n^{th}$ order vector finite element converges at the superconvergent rate of $O\left(\frac{|h/\lambda|^2}{n}\right)$ rather than the ordinary rate of $O\left(\frac{|h/\lambda|^n}{n}\right)$ [12].

The analysis presented here shows that second-order vector finite elements dramatically decrease the phase error. The use of second-order elements can dramatically increase the computational efficiency when accuracy is an issue. This is particularly true when the computational domain is electrically large since the phase error is cumulative [13]. For instance, looking at Figs. 3 and 6, one sees that to get the same error of 0.1 degrees per wavelength for any direction of propagation requires a value of $\lambda/h = 75$ for the first-order element but only a value of $\lambda/h = 16$ for the second-order element. The number of unknowns per square wavelength is approximately $(75/16)^2 \approx 22$ times larger for the first-order element as for the second-order element.

Finally, a simple approximation has been introduced that expresses the dispersion of the two-dimensional vector finite elements in terms of the dispersion of the one-dimensional nodal elements. The form of this approximation may work for elements of order greater than second but that has not been verified.

The conclusions made from this study are based on the quadrilateral vector element. Other types of vector elements may perform differently. For instance, in a recent numerical study of three-dimensional elements, tetrahedral elements are shown to be more accurate for a particular problem than hexagonal elements [14].

APPENDIX

The basis functions for the second order vector finite element are

\[
N_{1}^{\ell} = \alpha_{1}^{\ell}(y)\alpha_{2}^{\ell}(x)\hat{a}_{y} \quad \ell = 1, 2, 3 \\
N_{2}^{\ell+3} = \alpha_{2}^{\ell}(y)\alpha_{2}^{\ell}(x)\hat{a}_{y} \quad \ell = 1, 2, 3 \\
N_{3}^{\ell+6} = \alpha_{1}^{\ell}(x)\alpha_{2}^{\ell}(y)\hat{a}_{x} \quad \ell = 1, 2, 3 \\
N_{4}^{\ell+9} = \alpha_{1}^{\ell}(x)\alpha_{2}^{\ell}(y)\hat{a}_{x} \quad \ell = 1, 2, 3
\]

where the interpolation functions are

\[
\alpha_{1}^{\ell}(x) = \frac{\Delta x - x}{\Delta x} \\
\alpha_{2}^{\ell}(x) = \frac{x}{\Delta x} \\
\alpha_{3}^{\ell}(x) = \frac{2(x - \Delta x)\Delta x - x}{\Delta x^2} \\
\alpha_{4}^{\ell}(x) = \frac{4\Delta x(x - \Delta x)}{\Delta x^2} \\
\alpha_{5}^{\ell}(x) = \frac{2(x - \Delta x)}{\Delta x^2} .
\]
REFERENCES


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